SMOOTHNESS UP TO THE BOUNDARY
FOR SOLUTIONS OF THE NONLINEAR
AND NONELLIPTIC DIRICHLET PROBLEM

C. J. XU AND C. ZUILY

ABSTRACT. For the Dirichlet problem associated with a general real second order p.d.e. $F(x, u, \nabla u, \nabla^2 u) = 0$ in a smooth open set Ω of $\mathbf{R}^n$, we prove smoothness up to the boundary of the solution u for which the linearized characteristic form is nonnegative and satisfies Hörmander's brackets condition, the boundary of Ω being noncharacteristic.

0. Introduction. Let Ω be an open set in $\mathbf{R}^n$ with a C^∞ boundary $\partial\Omega$. Let F be a C^∞ real function in $\bar\Omega \times \mathbf{R}^N$ and φ be in $C^\infty(\partial\Omega)$. Let us consider a real solution u, in the Hölder space $C^\rho(\bar\Omega)$, of the problem

$$(0.1) \qquad \begin{cases} F(x, u(x), \nabla u(x), \nabla^2 u(x)) = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = \varphi. \end{cases}$$

We shall denote by $L_0(x, \xi)$ the principal symbol of the linearized form on u of equation (0.1), more precisely

$$(0.2) \qquad L_0(x, \xi) = \sum_{i,j=1}^{n} \frac{\partial F}{\partial u_{ij}}(x, u(x), \nabla u(x), \nabla^2 u(x))\xi_i \xi_j$$

and we set

$$L_0^{(j)}(x, \xi) = \frac{\partial L_0}{\partial \xi_j}(x, \xi), \qquad 1 \le j \le n.$$

This paper is mainly devoted to the proof of the following result.

THEOREM 0.1. *Let r be an integer and u be in $C^\rho(\bar\Omega)$ where $\rho > \mathrm{Max}(5, r+3)$. Let us suppose that*
(a) $L_0(x, \xi) \ge 0 \ \forall(x, \xi) \in \bar\Omega \times \mathbf{R}^n$.
(b) *If we denote by $\mathcal{L}_r$ the set of brackets of the vector fields $L_0^{(j)}$ of order less than or equal to r, then at each point of $\bar\Omega$ we can find n elements in $\mathcal{L}_r$ which are linearly independent.*
(c) $\partial\Omega$ *is noncharacteristic for $L_0(x, D)$.*
Then u belongs to $C^\infty(\bar\Omega)$.

The interior regularity of the solution has been proved in [8] as soon as $\rho > \mathrm{Max}(4, r + 2)$. In the elliptic case it is a classical fact that every $C^2(\bar\Omega)$ solution of such a Dirichlet problem is $C^\infty(\bar\Omega)$.

However, in our case, condition (b) by itself needs the solution to be at least $C^{r+2}(\bar\Omega)$; on the other hand, the condition $p > 5$ comes from the technique (mainly

---

the paradifferential Bony's theory). An attempt to replace condition (b) by a geometric condition which needs less a priori regularity of the solution, has been made by C. J. Xu [9]. But the optimality of the lower bound for ρ in the case of a completely nonlinear equation is still an open question. (See Remark 1.2 in C. Zuily [10].)

In the case of linear equations, the analogue of our result has been proved in various cases by J. J. Kohn and L. Nirenberg [4], M. Derridj [3], O. A. Oleinik and E. V. Radkevitch [6].

As we said before, the proof begins, following the ideas of J. M. Bony [2], by a tangential paralinearization (in M. Sablé-Tougeron's sense [7]). The mixed terms of the paradifferential equation are then eliminated, following S. Alinhac [1], by the paracomposition's technique (but not in the same spaces as in [1]). The end of the proof is more straightforward. We prove classical a priori subelliptic estimates using the symbolic calculus of the paradifferential operators, to get the tangential regularity and we use the equation to conclude.

The plan of the paper is as follows:

1. The paradifferential calculus.

1.1. *Tangential paralinearization.* We shall work in the spaces $H^{s,s'}(\bar{\mathbf{R}}^n_+)$ introduced by L. Hörmander. First for s and s' in $\mathbf{R}$ we set

$$H^{s,s'}(\mathbf{R}^n) = \{u \in S'(\mathbf{R}^n) \colon (1 + |\xi|^2)^{s/2}(1 + |\xi'|^2)^{s'/2}\hat{u} \in L^2(\mathbf{R}^n)\}.$$

Here $\xi = (\xi', \xi_n) \in \mathbf{R}^n$ and $\xi' \in \mathbf{R}^{n-1}$, the norm of this space being the natural one. Then $H^{s,s'}(\bar{\mathbf{R}}^n_+)$ is the space of restriction to $\bar{\mathbf{R}}^n_+ = \{x = (x', x_n) \colon x_n \geq 0\}$ of elements in $H^{s,s'}(\mathbf{R}^n)$. For $m \in \mathbf{R}$ and $\rho \in \mathbf{R}^*_+ \backslash \mathbf{N}$, Σ^m_ρ will denote the set of symbols of paradifferential operators defined in Bony [2]. If $\rho \in]0, 1[$ we set

$$C^\rho(\mathbf{R}^n) = \left\{u \in L^\infty(\mathbf{R}^n); [u]_\rho = \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\rho} < +\infty\right\}$$

with norm

$$\|u\|_{C^\rho} = \|u\|_{L^\infty} + [u]_\rho$$

and if $\rho = \rho_0 + k$, $0 < \rho_0 < 1$, $k \in \mathbf{N}$ $u \in C^\rho(\mathbf{R}^n) \Leftrightarrow D^\alpha u \in C^{\rho_0}$, $|\alpha| \leq k$.

Then $C^\rho(\bar{\mathbf{R}}^n_+)$ is the space of restriction to $\bar{\mathbf{R}}^n_+$ of elements in $C^\rho(\mathbf{R}^n)$. Let φ be a radial and positive function in $C^\infty_0(\mathbf{R}^n)$ which is equal to one for $|\xi| \leq \frac{1}{2}$ and vanishes for $|\xi| \geq 1$. We set $\tilde{\varphi}(\xi') = \varphi(\xi', 0)$. Following M. Sablé-Tougeron [7] we set for $u \in S'(\mathbf{R}^n)$ and p, p' in $\mathbf{N}$:

$$\widehat{S_p u}(\xi) = \varphi(2^{-p}\xi)\hat{u}(\xi), \qquad \widehat{S'_{p'} u}(\xi) = \tilde{\varphi}(2^{-p'}\xi')\hat{u}(\xi)$$

then $S_{pp'} = S_p \circ S'_{p'}$. We shall also set $\Delta_p = S_{p+1} - S_p$, $\Delta'_{p'} = S'_{p'+1} - S'_{p'}$ and $\Delta_{pp'} = \Delta_p \circ \Delta_{p'}$ which are also defined by

$$\widehat{\Delta_p u}(\xi) = \psi(2^{-p}\xi)\hat{u}(\xi), \qquad \widehat{\Delta'_{p'} u}(\xi) = \tilde{\psi}(2^{-p'}\xi')\hat{u}(\xi)$$

where $\psi(\xi) = \varphi(\xi/2) - \varphi(\xi)$, $\tilde{\psi}(\xi') = \psi(\xi', 0)$. So we have

$$\varphi(\xi) + \sum_{p \geq 0} \psi(2^{-p}\xi) = \tilde{\varphi}(\xi') + \sum_{p' \geq 0} \tilde{\psi}(2^{-p'}\xi') = 1.$$

Now if a and u are in $S(\mathbf{R}^n)$ we can define as in [7] the tangential paraproduct $\Pi'_a u$ and the paraproduct with two indices $\Pi''_a u$:

$$(1.1) \qquad\qquad \Pi'_a u = \sum_{p \geq N} S'_{p-N} a \Delta'_p u$$

$$(1.2) \qquad\qquad \Pi''_a u = \sum_{p,p' \geq N} S_{p-N,p'-N} a \cdot \Delta_{pp'} u.$$

The main result of this section is the following.

THEOREM 1.1. *Let F be a C^∞ function in $\bar{\mathbf{R}}^n_+ \times \mathbf{R}^d$ which is real and with compact support with respect to the x variable in $\bar{\mathbf{R}}^n_+$. Let $u_1, \ldots, u_d$ be real functions belonging to $C^\rho(\bar{\mathbf{R}}^n_+) \cap H^{m,s'}(\bar{\mathbf{R}}^n_+)$ where $m \in \mathbf{N}^*$, $\rho > m$, $s' > 0$. Then*

$$(1.3) \quad F(x, u_1(x), \ldots, u_d(x)) - \sum_{j=1}^d \Pi'_{(\partial F/\partial u_j)(x, u_1(x), \ldots, u_d(x))} u_j \in H^{m,s'+\rho}(\bar{\mathbf{R}}^n_+).$$

PROOF. As in [8, Proposition 1.3] we can reduce to the case where F does not depend on x. Moreover, taking restrictions to $\mathbf{R}^n_+$, the above result will be a consequence of the same result in the spaces $H^{m,s'}(\mathbf{R}^n)$. On the other hand we do not decrease the generality if we suppose $d = 1$. We shall set $u_1 = u$.

Since $u \in C^\rho(\mathbf{R}^n)$, $\rho > 0$ we have $\lim_{q \to +\infty} \|S'_q u - u\|_{L^\infty} = 0$. So following [5] we can write

$$(1.4) \qquad\qquad F(u) = F(S_0 u) + \sum_{q=0}^{+\infty} \{F(S'_{q+1} u) - F(S'_q u)\}.$$

Moreover for every N, $S'_N u \in H^{m,+\infty}(\mathbf{R}^n)$ and since for $m \geq 1$, $H^{m,+\infty}(\mathbf{R}^n)$ is an algebra [7, Proposition 1.7] we have $F(S'_N u) \in H^{m+\infty}(\mathbf{R}^n)$. Now

$$F(S'_{q+1} u) - F(S'_q u) = \Delta'_q u \int_0^1 F'(S'_q u + t\Delta'_q u)\, dt.$$

From (1.1) to (1.4) we get

$$F(u) - \Pi'_{F'(u)} u = F(S'_N u) + \sum_{q \geq N} \Delta'_q u \left\{ \int_0^1 F'(S'_q u + t\Delta'_q u)\, dt - S'_{q-N}(F'(u)) \right\}.$$

So we have to prove

$$(1.5) \quad g = \sum_{q \geq N} \Delta'_q u \left\{ \int_0^1 F'(S'_q u + t\Delta'_q u)\, dt - S'_{q-N}(F'(u)) \right\} \in H^{m,s'+\rho}(\mathbf{R}^n).$$

Cutting each term $\Delta'_q u$ we get

$$g = \sum_{q \geq N} \sum_{q \leq p+3} \Delta_p \Delta'_q u \left\{ \int_0^1 F'(S'_q + t\Delta'_q u) \, dt - S'_{q-N}(F'(u)) \right\}$$

$$= \sum_{p,q} a_{pq}$$

Indeed if $q \geq p+4$, $\{\frac{1}{2} 2^p \leq |\xi| \leq 2^{p+2}\} \cap \{\frac{1}{2} 2^q \leq |\xi| \leq 2^{q+2}\} = \emptyset$.

We shall prove that

(1.6)
$$\forall \alpha \in \mathbf{N}^n, \ |\alpha| \leq m, \ \forall \alpha' \in \mathbf{N}^{n-1}$$
$$\|D_x^\alpha D_{x'}^{\alpha'} a_{pq}\|_{L^2} \leq C_{pq} 2^{p(|\alpha|-m)} 2^{q(|\alpha'|-s'-\rho)}, \qquad (C_{pq}) \in l^2.$$

First, since $u \in H^{m,s'}(\mathbf{R}^n)$ we have

(1.7)
$$\|D_{x_n}^{\alpha_n} D_{x'}^{\alpha'} \Delta_p \Delta'_q u\|_{L^2} \leq C_{pq} 2^{p(\alpha_n - m)} 2^{q(|\alpha'|-s')}, \qquad (C_{pq}) \in l^2.$$

On the other hand if $\alpha_n \leq m < \rho$, $D_{x_n}^{\alpha_n} u \in C^{\rho - \alpha_n}(\mathbf{R}^n)$ so

$$D_{x_n}^{\alpha_n} F'(u) \in C^{\rho - \alpha_n}(\mathbf{R}^n)$$

from which we deduce

(1.8)
$$\|D_{x_n}^{\alpha_n} D_{x'}^{\alpha'} \{ S'_{q-N}(F'(u)) - F'(u) \}\|_{L^\infty}$$
$$= \|D_{x'}^{\alpha'} \{ S'_{q-N}(D_{x_n}^{\alpha_n} F'(u)) - D_{x_n}^{\alpha_n} F'(u) \}\|_{L^\infty}$$
$$\leq C 2^{q(|\alpha'|-(\rho - \alpha_n))}.$$

Now, if $\alpha_n \leq m$

(1.9)
$$\left\| D_{x_n}^{\alpha_n} D_{x'}^{\alpha'} \left\{ \int_0^1 F'(S'_q u + t\Delta'_q u) \, dt - F'(u) \right\} \right\|_{L^\infty} \leq C 2^{q(|\alpha'|-(\rho - \alpha_n))}.$$

Indeed the left-hand side of (1.9) is bounded by

$$A = \sup_{0 \leq t \leq 1} \|D_{x_n}^{\alpha_n} D_{x'}^{\alpha'} \{ F'(S'_q u + t\Delta'_q u) - F'(u) \}\|_{L^\infty}.$$

Now

$$F'(S'_q u + t\Delta'_q u) - F'(u) = (S'_q u - u + t\Delta'_q u) F''(\theta S'_q u + \theta t \Delta'_q u + (1-\theta)u).$$

We easily get

(1.10)
$$\|D_{x_n}^{\beta_n} D_{x'}^{\beta'} \{ (S'_q u - u + t\Delta'_q u) \}\|_{L^\infty} \leq C_0 2^{q(|\beta'|-(\rho - \alpha_n))}.$$

and using Lemma 3 in [5] we get

(1.11)
$$\|D_{x_n}^{\gamma_n} D_{x'}^{\gamma'} F''(\theta S'_q u + \theta t \Delta'_q u + (1-\theta)u)\|_{L^\infty} \leq C 2^{q(|\gamma'|+|\gamma_n|)}$$

and (1.9) follows easily from (1.10) and (1.11).

From (1.7), (1.8) and (1.9) we get

$$\|D_{x_n}^{\alpha_n} D_{x'}^{\alpha'} a_{pq}\|_{L^2} \leq C_{pq} 2^{p(\alpha_n - m)} 2^{q(|\alpha'|-s'-\rho)}, \qquad (C_{pq}) \in l^2,$$

which implies (1.6).

Now Theorem 1.1 follows from the following lemma (see Lemma 2.3 in [7]).

LEMMA 1.2. *Let $(a_{pq})_{q<p}$ be a sequence of functions such that for $\alpha_n \leq m$, $\alpha' \in \mathbf{N}^{n-1}$,*

(1.12) $$\|D_{x_n}^{\alpha_n} D_{x'}^{\alpha'} a_{pq}\|_{L^2} \leq C_{pq\alpha} 2^{p(\alpha_n - m) + q(|\alpha'| - t')}$$

where $m \in \mathbf{N}$, $t' \in \mathbf{R}^+$ and $(C_{pq\alpha}) \in l^2$.
  Then $g = \sum_{q<p} a_{pq} \in H^{m,t'}(\mathbf{R}^n)$.

1.2 *Paracomposition.* Let $\Omega_j = \omega_j \times [0, T_j[$, $j = 1, 2$ be two open sets in $\bar{\mathbf{R}}_+^n = \{(x', x_n): x_n \geq 0\}$ and χ be a C^σ-diffeomorphism from Ω_1 to Ω_2 such that $\sigma > 3$ and

(1.13) $\quad \chi(y', y_n) = (x', x_n)$ with $x_j = f_j(y', y_n)$, $1 \leq j \leq n-1$, $x_n = y_n$.

We shall denote by $C_{\text{loc}}^\alpha(\omega \times [0, T[)$ the space of distributions on $\omega \times [0, T[$ such that ςu belongs to $C^\alpha(\bar{\mathbf{R}}_+^n)$ for all ς in $C^\infty(\bar{\mathbf{R}}_+^n)$ with supp $\varsigma = K \times [0, \varepsilon]$ where K is compact in $\omega \subset \mathbf{R}^{n-1}$ and $\varepsilon < T$. The space $H_{\text{loc}}^{s,s'}(\omega \times [0, T[)$ is then defined in the same way.
  Following S. Alinhac [1] we now prove

THEOREM 1.3. *There exists a linear operator $\chi^*: \mathcal{D}'(\Omega_2) \to \mathcal{D}'(\Omega_1)$ with the following properties:*
  (i) *χ^* is continuous from $C_{\text{loc}}^\alpha(\Omega_2) \cap H_{\text{loc}}^{s,s'}(\Omega_2)$ to $C_{\text{loc}}^\alpha(\Omega_1) \cap H_{\text{loc}}^{s,s'}(\Omega_1)$ for all strictly positive α, s and s'.*
  (ii) *Let u be in $H_{\text{loc}}^{m,t}(\Omega_2)$, $m \geq 1$, $t \geq 0$, and $P' = \sum_{|\alpha| \leq 2} \Pi_{a_\alpha}' \partial_x^\alpha \in \mathrm{Op}(\Sigma_\sigma^2)$ with $a_{nn} = 1$, be a paradifferential operator such that $P'u \in H_{\text{loc}}^{m,t}(\Omega_2)$. Then if $\sigma > \mathrm{Max}(3, m+1)$ we have*

$$\chi^*(P'u) - (P^*)'\chi^*u \in H^{m,t+\sigma-3-\varepsilon}(\Omega_1) \quad \forall \varepsilon > 0$$

where $(P^*)' = \sum_{|\alpha| \leq 2} \Pi_{b_\alpha}' \partial_y^\alpha \in \mathrm{Op}(\Sigma_{\sigma-1}^2)$ and the principal symbol of $(P^*)'$ is

$$p^*(y, \eta) = \sum_{|\alpha|=2} a_\alpha(\chi(y)) \left({}^t \left(\frac{\partial \chi}{\partial x} \right)^{-1} (\chi(y)) \eta \right).$$

  (iii) *Let $\chi_0: \Omega_0 \to \Omega_1$ and $\chi_1: \Omega_1 \to \Omega_2$ be two C^σ-diffeomorphisms as in (1.13). Then $\chi_0^*\chi_1^*u = (\chi_1\chi_0)^*u + Ru$ where R is continuous from $H^{s,s'}$ to $H^{s,s'+\sigma-1}$.*

We shall consider two systems of coronas: the small coronas

$$C_p = \{\xi \in \mathbf{R}^n: c^{-1}2^p \leq |\xi| \leq c2^{p+1}\},$$
$$C_p' = \{\xi = (\xi', \xi_n) \in \mathbf{R}^n: c^{-1}2^p \leq |\xi'| \leq c2^{p+1}\}$$

with partition of unity $1 = \varphi(\xi) + \sum_{p=1}^\infty \psi(2^{-p}\xi)$; we set $\delta_p = \psi(2^{-p}D)$, $\delta_p' = \psi(2^{-p}D', 0)$ and $\delta_{pp'} = \delta_p \circ \delta_{p'}'$; and the big coronas

$$C_p = \{\xi \in \mathbf{R}^n: C^{-1}2^P \leq |\xi| \leq C2^{P+1}\},$$
$$C_p' = \{\xi = (\xi', \xi_n) \in \mathbf{R}^n: C^{-1}2^P \leq |\xi'| \leq C2^{P+1}\}$$

with the partition of unity $1 = \Phi(\xi) + \sum_{p=1}^{+\infty} \Psi(2^{-p}\xi)$; we set $\Delta_p = \Psi(2^{-p}D)$ $\Delta_p' = \Psi(2^{-p}D', 0)$ and $\Delta_{pp'} = \Delta_p \circ \Delta_p'$.

Let us note that $C_p \cap C_{p'} = \varnothing$ when $p' > p + 1$.

The distributions $u \in \mathcal{E}'(\Omega_2)$ will be cut, using the small coronas

$$u = \sum_{p' \leq p+1} u_{pp'}.$$

Given a compact K in Ω_2 we choose the constant C such that

(1.14)   For every y in a neighborhood of $\chi^{-1}(K)$
              and every ξ in c_p we have $^t(\partial\chi(y)/\partial y)\xi \in C_p$.

The distributions $v \in \mathcal{E}'(\Omega_1)$ will be cut using the big coronas. We set

$$[v]_{pp'} = \sum_{\substack{|j-p| \leq N \\ |j'-p'| \leq N}} \Delta_{jj'} v$$

which means that the sum is taken over (j, j') such that the spectrum of $\Delta_{jj'}v$ meets $C_p \cap C'_{p'}$. Here N is an integer related to the constant C.

DEFINITION 1.4. Let u be in $\mathcal{E}'(\Omega_2)$ with $\operatorname{supp} u \subset K$. We set

(1.15) $$\chi^*(u) = \sum_{p' \leq p+1} [\psi(u_{pp'} \circ \chi(y)]_{pp'}$$

where $\psi \in C_0^\infty(\Omega_1)$, $\psi = 1$ near $\chi^{-1}(K)$.

PROOF OF THEOREM 1.3. (i) This part is just an application of Lemma 1.6 in [7].

(ii) We first give two lemmas. We take $\psi_1 \in C_0^\infty(\Omega_1)$, $\psi_1 = 1$ near the support of ψ and we set

$$S_{pp'}\chi = \sum_{\substack{q \leq p \\ q' \leq p'}} (\Delta_{qq'}\psi_1\chi', \chi_n), \qquad \chi' = (\chi_1, \ldots, \chi_{n-1}).$$

LEMMA 1.5. *Let u be in $\mathcal{E}'(\Omega_2)$, $\operatorname{supp} u \subset K$. Let us set*

(1.16) $$Ru = \chi^*(u) - \sum_{p' \leq p+1} [\psi(u_{pp'} \circ S_{pp'}\chi(y))]_{pp'}.$$

Then R maps $H^{s,s'}(\Omega_2)$ in $H^{s,s'+\sigma-1}(\Omega_1)$ for $s > 0$, $s' > 0$.

PROOF. Since $\chi \in C^\sigma$ we have

$$||\psi_1\chi' - S_{pp'}\psi_1\chi'||_{L^\infty} \leq ||\psi_1\chi' - S_p\psi_1\chi'||_{L^\infty} + ||S_p(\psi_1\chi' - S'_{p'}\psi_1\chi')||_{L^\infty}$$
$$\leq C2^{-p\sigma} + C2^{-p'\sigma} \leq C_1 2^{-p'\sigma}.$$

Now, since $x_n = y_n$, for $u \in H^{s,s'}$ we have

$$||[\psi(u_{pp'} \circ \chi) - \psi(u_{pp'} \circ S_{pp'}\chi]_{pp'}||_{L^2} \leq ||u_{pp'} \circ \chi - u_{pp'} \circ S_{pp'}\chi||_{L^2}$$
$$\leq C_0||D_{x'}u_{pp'}||_{L^2}||\psi_1\chi' - S_{pp'}\psi_1\chi'||_{L^\infty}$$
$$\leq C_{pp'}2^{-ps-p'(s-1)-p'\sigma} \leq C_{pp'}2^{-ps-p'(s+\sigma-1)}$$

with $(C_{pp'}) \in l^2$. It follows that $Ru \in H^{s,s'+\sigma-1}$.

LEMMA 1.6. *Let u be in $H^{s,s'}(\Omega_2)$ $s > 0$, $s' > 0$. Then*

$$v = \sum_{p,p'} [\psi(u_{pp'} \circ S_{pp'}\chi)]_{pp'} - \sum_{p,p'} \psi(u_{pp'} \circ S_{pp'}\chi) \in H^{s,s'+\sigma-1}.$$

PROOF. We can write

$$\sum_{pp'} \psi(u_{pp'} \circ S_{pp'}\chi) = \sum_{p,p'} [\psi(u_{pp'} \circ S_{pp'}\chi)]_{pp'}$$

$$+ \sum_{p,p'} \left\{ \sum_{\substack{|q-p|>N \\ q'}} + \sum_{\substack{|q-p|\leq N \\ |q'-p'|>N}} \Delta_{qq'}(\psi(u_{pp'} \circ S_{pp'}\chi)) \right\}.$$

Now, from the Lemma in §2.1.2 of [1], we get

$$\left\| \sum_{\substack{|q-p|>N \\ q'}} \Delta_{qq'}(\psi(u_{pp'} \circ S_{pp'}\chi)) \right\|_{L^2} \leq C_{pp'} 2^{-p(s+\sigma-1)-p's'}, \qquad (C_{pp'}) \in l^2,$$

$$\left\| \sum_{\substack{|q-p|\leq N \\ |q'-p'|>N}} \Delta_{qq'}(\psi(u_{pp'} \circ S_{pp'}\chi)) \right\|_{L^2} \leq C_{pp'} 2^{-ps} 2^{-p(s'+\sigma-1)}, \qquad (C_{pp'}) \in l^2,$$

which prove Lemma 1.6.

Now, as in [1] if we change the coronas or the functions ψ, ψ_1, the definition of χ^* is modified by operators which are $\sigma - 1$ smoothing. So we can extend the definition of χ^* to $\mathcal{D}'$.

Let us consider now a paradifferential operator in $\mathrm{Op}(\Sigma_\sigma^2)$

$$P' = D_n^2 + \sum_{j=1}^{n-1} \Pi'_{a_{jn}} D_j D_n + \sum_{i,j=1}^{n-1} \Pi'_{a_{ij}} D_i D_j + \sum_{|\alpha|\leq 1} \Pi'_{b_\alpha} D^\alpha$$

and let P'' be the same operator but with Π''_a instead of Π'_a.

LEMMA 1.7. *Let u be in $H^{m,t}(\bar{\mathbf{R}}_+^n)$ with $\sigma > m+1 \geq 2$ and $t > 0$. Then*

(1.17) $$P'u - P''u \in H^{m,t+\sigma-2-\varepsilon}(\bar{\mathbf{R}}_+^n) \quad \forall \varepsilon > 0.$$

PROOF. If we set $\tilde{P}' = P' - D_n^2$ then $P' - P'' = \tilde{P}' - \tilde{P}''$. Now $\tilde{P}'u = \sum_\alpha \sum_{p'} S'_{p'-N}(a_\alpha) \partial^\alpha u_{p'}$ and

$$S'_{p'-N}(a_\alpha) \partial^\alpha u_{p'} = \sum_{|p-q|\leq N} \delta_q(S_{p'-N}(a_\alpha)) \delta_p(\partial^\alpha u_{p'})$$

$$+ \sum_{|p-q|\geq N} \delta_q(S'_{p'-N}(a_\alpha)) \delta_p(\partial^\alpha u_{p'})$$

from which we deduce

$$\tilde{P}'u = \sum_\alpha \sum_{p,p'} S_{p-N,p'-N}(a_\alpha)\partial^\alpha u_{pp'}$$

$$\text{①}$$

$$+ \sum_\alpha \sum_{p,p'} \delta_p S'_{p-N}(a_\alpha)\partial^\alpha S_{p-N}u_{p'}$$

$$\text{②}$$

$$+ \sum_\alpha \sum_{\substack{|p-q|\leq N \\ p'}} \delta_q S'_{p'-N}(a_\alpha)\partial^\alpha u_{pp'}$$

$$\text{③}$$

First of all we have

$$\text{①} = \tilde{P}''u.$$

Now, since in the sum $|\alpha| \leq 2$ and $\alpha \neq (n,n)$ we have $\partial^\alpha u \in H^{0,m+t-|\alpha|}$ so

$$\|\partial^\alpha S_{p-N}u_{p'}\|_{L^2} = \|S_{p-N}\partial^\alpha u_{p'}\|_{L^2} \leq C_{p'}2^{-p'(m+t-|\alpha|)}, \qquad (C_{p'}) \in l^2,$$
$$\|\delta_p S'_{p-N}(a_\alpha)\|_{L^\infty} \leq C2^{-p(\sigma+|\alpha|-2)} \leq C2^{-p\varepsilon}2^{-p(\sigma+|\alpha|-2-\varepsilon)}.$$

For $\sigma + |\alpha| - 2 - m - \varepsilon \geq 0$ we have ② $\in H^{\sigma+|\alpha|-2-\varepsilon,m+t-|\alpha|} \subset H^{m,t+\sigma-2-\varepsilon}$. The same argument can be used for ③.

LEMMA 1.8. *Let u be in $H^{m,t}(\bar{\mathbf{R}}^n_+)$ such that $P''u$ is in $H^{m,t}(\bar{\mathbf{R}}^n_+)$. If $\sigma > m + 1 \geq 2$ and $t > 0$, we have*

$$(1.18) \qquad \chi^*(P''u) - P^{*''}(\chi^*u) \in H^{m,t+\sigma-3}(\bar{\mathbf{R}}^n_+).$$

PROOF.

$$\chi^*(P''u) = \sum_\alpha \sum_{p,p'} [(S_{p-N,p'-N}(a_\alpha)\partial_x^\alpha u_{pp'}) \circ S_{pp'}\chi(y)]_{pp'}$$

$$= \sum_\alpha \sum_{p,p'} (S_{p-N,p'-N}(a_\alpha) \circ S_{pp'}\chi)\left({}^t\left(\frac{\partial S_{pp'}\chi(y)}{\partial y}\right)\partial_y\right)^\alpha (u_{pp'} \circ S_{pp'}\chi) + Ru$$

where $Ru \in H^{m,t+\sigma-1}$ by Lemma 1.6.
On the other hand,

$$P^{*''}(\chi^*u) = \sum_\alpha \sum_{p,p'} \left(S_{p-N,p'-N}\left(a_\alpha \circ \chi\left({}^t\left(\frac{\partial\chi}{\partial y}(y)\right)\partial_y\right)^\alpha\right)\right)[u_{pp'} \circ S_{pp'}\chi]_{pp'}$$

$$= \sum_\alpha \sum_{p,p'} S_{p-N,p'-N}(\tilde{a}_\alpha)\partial_y^\alpha[u_{pp'} \circ S_{pp'}\chi]_{pp'}.$$

We have $P''u = D_n^2u + \Pi''_{\tilde{p}(x,\xi)}u \in H^{m,t}$ so $D_n^2u = f + g$ where $f \in H^{m,t}$ and $g \in H^{m-1,t-1}$. Therefore $u \in H^{m+1,t-1}$ so $g \in H^{m,t-2}$ and $u \in H^{m+2,t-2}$.
Now for $\chi \in C^\sigma$, $a \in C^\sigma$, $b \in C^{\sigma-1}$ let us consider

$$A = \|S_{p-N,p'-N}((a \circ \chi)b) - S_{p-N,p'-N}(a) \circ S_{pp'}\chi S_{pp'}b\|_{L^\infty}.$$

We have

$$\|S_{p-N,p'-N}(a \circ \chi)S_{pp'}b - S_{p-N,p'-N}(a) \circ S_{pp'}\chi S_{pp'}b\|_{L^\infty}$$
$$\leq C\|S_{p-N,p'-N}(a \circ \chi) - S_{p-N,p'-N}(a) \circ S_{pp'}\chi\|_{L^\infty}$$
$$\leq C\{\|S_{p-N,p'-N}(a \circ \chi) - a \circ \chi\|_{L^\infty} + \|(a - S_{p-N,p'-N}(a)) \circ \chi\|_{L^\infty}$$
$$+ \|S_{p-N,p'-N}(a) \circ \chi - S_{p-N,p'-N}(a) \circ S_{pp'}\chi\|\}.$$
$$\leq C2^{-p'\sigma}$$

On the other hand,

$$\|S_{p-N,p'-N}((a \circ \chi)b) - S_{p-N,p'-N}(a \circ \chi)S_{pp'}b\|_{L^\infty}$$
$$\leq \|S_{p-N,p'-N}((a \circ \chi)b) - (a \circ \chi)b\|_{L^\infty}$$
$$+ \|(a \circ \chi - S_{p-N,p'-N}(a \circ \chi))b\|_{L^\infty}$$
$$+ \|S_{p-N,p'-N}(a \circ \chi)(b - S_{pp'}b)\|_{L^\infty}$$
$$\leq C2^{-p'(\sigma-1)}.$$

So we have

(1.19) $$A \leq C2^{-p'(\sigma-1)}.$$

Now, since u belongs to $H^{m+2,t-2}$, Lemma 1.6 implies that

$$\|\partial^\alpha((u_{pp'} \circ S_{pp'}\chi) - [u_{pp'} \circ S_{pp'}\chi]_{pp'})\|_{L^2} \leq C_{pp'}2^{-pm-p'(t+\sigma-3)}.$$

Now

$$\chi^*(P''u) - P^{*''}(\chi^*u) = \chi^*(P''u) - \sum_\alpha \sum_{p,p'} S_{p-N,p'-N}(\tilde{a}_\alpha)\partial_y^\alpha(u_{pp'} \circ S_{pp'}\chi)$$

$$+ \sum_\alpha \sum_{p,p'} S_{p-N,p'-N}(\tilde{a}_\alpha)\partial_y^\alpha(u_{pp'} \circ S_{pp'}\chi)$$

$$- P^{*''}(\chi^*u) \in H^{m,t+\sigma-3}. \quad \text{Q.E.D.}$$

To prove (ii) in Theorem 1.3 we write

$$\chi^*(P'u) - P^{*'}(\chi^*u) = \chi^*(P'u - P''u) + \chi^*(P''u) - P^{*''}(\chi^*u) + (P^{*''} - P^{*'})(\chi^*u)$$

and we apply Lemmas 1.7 and 1.8.

Let us now prove (iii) in Theorem 1.3. To define χ_1^* we use the coronas adapted to χ_1 and K as in (1.15). So $\chi_1^*u = \sum_{p' \leq p+1}[\varsigma(u_{pp'} \circ \chi_1)]_{pp'}$ with $\psi = 1$ near supp ς. To define χ_0^* we take a system of small coronas such that $[\ ]_{pp'}$ has its spectrum in $C_p \cap C_{p'}$ and a system of big coronas corresponding to the small system and to $\chi_1^{-1}(K)$. Let $\varsigma_0 \in C_0^\infty(\Omega)$ $\varsigma_0 = 1$ near $\chi_0^{-1}(\text{supp }\psi)$. We have

$$\chi_0^*\psi\chi_1^*u = \sum_{p' \leq p+1} [\varsigma_0([\varsigma(u_{pp'} \circ \chi_1)]_{pp'} \circ \chi_0)]_{pp'}$$

$$= \sum_{p' \leq p+1} [\varsigma_0(\varsigma(u_{pp'} \circ \chi_0))]_{pp'} + \sum_{p' \leq p+1} [\varsigma_0(R_{pp'} \circ \chi_0)]_{pp'}$$

where

$$R_{pp'} = [\varsigma(u_{pp'} \circ \chi_1)]_{pp'} - \varsigma(u_{pp'} \circ \chi_1).$$

Using Lemmas 1.5 and 1.6 we get for $u \in H^{s,s'}$

$$\|R_{pp'}\|_{L^2} \leq C_{pp'} 2^{-ps-p'(s'+\sigma-1)}$$

so $\sum_{p' \leq p+1} [\varsigma_0 (R_{pp'} \circ \chi_0)]_{pp'} \in H^{s,s'+\sigma-1}$. Now,

$$\varsigma_0 (\varsigma u_{pp'} \circ \chi_1 \circ \chi_0) = \varsigma_0 (\varsigma_1 \circ \chi_0)(u_{pp'} \circ (\chi_1 \circ \chi_0)) = \varsigma_2 (u_{pp'} \circ (\chi_1 \circ \chi_0))$$

where $\varsigma_2 = 1$ in a neighborhood of $(\chi_1 \chi_0)^{-1}(K)$. This proves (iii).

2. Proof of Theorem 0.1.

2.1 *Preliminaries.* We know by the Theorem 1 of [8] that the solution of (0.1) is C^∞ inside Ω so we have to prove now its smoothness near the boundary. Let x_0 be a point in $\partial \Omega$ and V a fixed neighborhood of x_0 in $\bar{\Omega}$. Let ς be in $C_0^\infty(\bar{V})$, $\varsigma = 1$ near x_0. By a C^∞-diffeomorphism Θ near x_0 we can reduce the problem to the case $x_0 = 0$ and $\partial \Omega \cap V = \{x = (x', x_n) \in \mathbf{R}^n : x_n = 0\}$. We set $v = \varsigma u \circ \Theta^{-1}$. Using the implicit function theorem, condition (c) in Theorem 0.1 and subtracting from v a C^∞ extension of the boundary condition we find that v is a solution of the problem.

$$(2.1) \qquad \begin{cases} \partial_n^2 v + G(x, v(x), \nabla v(x), \tilde{\nabla}^2 v(x)) = 0 & \text{in } V \times \bar{\mathbf{R}}_+^n, \\ v|_{x_n=0} = 0, \end{cases}$$

where $\tilde{\nabla}^2 v$ means $(\partial^\alpha v)$, $|\alpha| = 2$, $\alpha = (\alpha', \alpha_n)$, $\alpha_n < 2$. Conditions (a) and (b) obviously remain invariant. Applying Theorem 1.1 to our function G we get

PROPOSITION 2.1. *Let* $v \in C_{\text{loc}}^\rho(\bar{\mathbf{R}}_+^n) \cap H_{\text{loc}}^{s,t}(\bar{\mathbf{R}}_+^n)$, $\rho > s+2$, $\rho > 5$, $t \geq 1$, *be a solution of problem* (2.1). *Let us set*

$$P' = D_n^2 + \sum_{\substack{|\alpha| \leq 2 \\ \alpha_n < 2}} \Pi'_{a_\alpha} D^\alpha \quad \text{with } a_\alpha(x) = \frac{\partial G}{\partial u_\alpha}(x, v(x), \nabla v(x), \tilde{\nabla}^2 v(x)).$$

Then $P'v \in H_{\text{loc}}^{s,t+\rho-4}(\bar{\mathbf{R}}_+^n)$.

Indeed Theorem 1.1 gives $P'v = f \in H_{\text{loc}}^{s-1,t+\rho-3}(\bar{\mathbf{R}}_+^n)$. Now if v is in $H_{\text{loc}}^{s,t}$ then $\sum_{|\alpha| \leq 2; \alpha_n < 2} \Pi'_{a_\alpha} D^\alpha v$ belongs to $H_{\text{loc}}^{s-1,t-1}$ so $D_n^2 v \in H_{\text{loc}}^{s-1,t-1}$ and since v is in $H_{\text{loc}}^{s,t}$ we get $v \in H^{s+1,t-1}$ so, applying another time Theorem 1.1, we get $P'v \in H_{\text{loc}}^{s,t+\rho-4}$.

The principal symbol of P' is equal to

$$p'(x, \xi) = \xi_n^2 + 2 \sum_{j=1}^{n-1} \frac{\partial G}{\partial u_{jn}}(\cdots) \xi_j \xi_n + \sum_{ij=1}^{n-1} \frac{\partial G}{\partial u_{ij}}(\cdots) \xi_i \xi_j.$$

Let g_k, $1 \leq k \leq n-1$, be the solutions near the origin of the Cauchy problems

$$(2.2) \qquad \begin{cases} \dfrac{\partial g_k}{\partial x_n} + \sum_{j=1}^{n-1} \dfrac{\partial G}{\partial u_{jn}}(x, v(x), \nabla v(x), \tilde{\nabla}^2 v(x)) \dfrac{\partial g_k}{\partial x_j} = 0, \\ g_k|_{x_n=0} = x_k, \end{cases}$$

where v is a solution of (2.1) which is in $C_{\text{loc}}^\rho \cap H^{s,t}$. Then the solutions g_k exist locally and are in $C^{\rho-2}$ near the origin.

Let Θ be the $C^{\rho-2}$ local diffeomorphism defined by

$$\Theta(x', x_n) = (y', y_n) \quad \text{with} \quad \begin{cases} y_k = g_k(x), & 1 \le k \le n-1, \\ y_n = x_n, \end{cases}$$

and $\chi = \Theta^{-1}$. Applying Proposition 2.1, Theorem 1.3(i) and (ii) with $\sigma = \rho - 2 > \text{Max}(3, s+1)$ we get the following proposition.

PROPOSITION 2.3. *Let $v \in C^{\rho}_{\text{loc}} \cap H^{s,t}_{\text{loc}}$, where $\rho > \text{Max}(5, s+3)$, be a solution of (2.1). Then $(P^*)'(\chi^* v) \in H^{s,t+\rho-5-\varepsilon}_{\text{loc}}$ for every $\varepsilon > 0$.*

Now $(P^*)'$ is in $\text{Op}(\Sigma^2_{\rho-3})$ and its principal symbol is

$$(2.3) \qquad\qquad p^*(y, \eta) = \eta_n^2 + \sum_{i,j=1}^{n-1} a_{ij}(y)\eta_i\eta_j.$$

Moreover $p^*(y, \eta) \ge 0$ for all (y, η) in $V \times \mathbf{R}^n$ and condition (b) in Theorem 0.1 is also satisfied by the operator $(P^*)'$ if $\rho > \text{Max}(5, s+3)$.

We shall denote in the following Q and (x, ξ) instead of $(P^*)'$ and (y, η).

2.2 *The subelliptic estimate.* The main result of this section is the following

PROPOSITION 2.4. *For every compact K in $\bar{\mathbf{R}}^n_+$ we can find a positive constant C and $\varepsilon > 0$ such that*

$$\|u\|^2_{0,\varepsilon} \le C\{|(Qu, u)| + \|u\|^2_{0,0}\}$$

for every $u \in E = \{u \in C^\infty(\bar{\mathbf{R}}^n_+): \text{supp}\, u \subset K, \ u_{|x_n=0} = 0\}$.

PROOF. Since the proof is classical and very close to the proof of Proposition 1.10 in [8] we shall only sketch it.

Let $q_0(x, \xi)$ be the principal symbol of Q. For $1 \le j \le n$ we denote by Q_j (resp. Q_{j+n}) the paradifferential operator with symbol

$$\partial q_0/\partial \xi_j \quad (\text{resp. } \varphi(1 + |\xi'|^2)^{-1/2}\partial q_0/\partial x_j).$$

We prove first

$$(2.5) \qquad\qquad \sum_{j=1}^{2n} \|Q_j u\|^2_{0,0} \le C\{|(Qu, u)| + \|u\|^2_{0,0}\} \quad \forall u \in E.$$

Writing

$$q(x, \xi) = \xi_n^2 + \sum_{i,j-1}^{n-1} a_{ij}(x)\xi_i\xi_j$$

where $a_{ij} \in C^{\rho-2}$ we know that $\sum_{i,j=1}^{n-1} a_{ij}(x)\xi_i\xi_j \ge 0$ for every (x, ξ) in $V \times \mathbf{R}^n$. So we can use Lemma 1.7.1 of [6] and the corollary which follows it to write

$$(2.6) \quad \begin{cases} |\sum_{i,j=1}^{n-1} \frac{\partial}{\partial x_\alpha} a_{ij}(x)\partial_i\partial_j u(x)|^2 \le M \sum_{i,j,s=1}^{n-1} a_{ij}(x)\partial_j\partial_s u(x)\partial_i\partial_s u(x), \\ |\sum_{k=1}^{n-1} a_{kj}(x)\partial_k u(x)|^2 \le 2a_{jj}(x) \sum_{k,j=1}^{n-1} a_{kj}(x)\partial_j u(x)\partial_k u(x). \end{cases}$$

Here the constant M depends only on the second derivative of the coefficients on K. If we denote by G_j the vector fields

$$\sum a_{kj} \frac{\partial}{\partial x_k} \qquad 1 \le j \le n-1 \quad \text{and} \quad G_n = D_n$$

we deduce from (2.6)

$$(2.7) \qquad \sum_{j=1}^{n} \|G_j u\|_{0,0}^2 \leq C \left\{ \|D_n u\|^2 + \sum_{j,k=1}^{n-1} (a_{kj} \partial_j u, \partial_k u) + \|u\|_{0,0}^2 \right\}.$$

Now we have

LEMMA 2.5. Let $P = \sum_{|\alpha| \leq m} a_\alpha(x', x_n) D_{x'}^\alpha$ where $a_\alpha \in C^{\tilde{\rho}}$ with compact support in $\bar{\mathbf{R}}_+^n$ and $\tilde{\rho} > m$. Then $P - \sum_{|\alpha| \leq m} \Pi'_{a_\alpha} D_x^\alpha$ is bounded from $L^2(\mathbf{R}_+^n)$ to $L^2(\mathbf{R}_+^n)$.

This lemma follows from Theorem 3.4 in [2] if we consider x_n as a parameter. Now (2.5) follows from (2.7) and Lemma 2.5.

The next step is the proof of the estimate

$$(2.8) \qquad \|u\|_{0,\varepsilon}^2 \leq C \left\{ \sum_{j=1}^{n} \|Q_j u\|_{0,0}^2 + \|u\|_{0,0}^2 \right\}$$

for every u in the space E.

The proof follows classical ideas (see [6]). We write

$$\|u\|_{0,\varepsilon}^2 \leq C \left\{ \sum_{j=1}^{n-1} \left\| \frac{\partial u}{\partial x_i} \right\|_{0,\varepsilon-1}^2 + \|u\|_{0,\varepsilon-1}^2 \right\}.$$

Using condition (b) in Theorem 0.1 we get

$$\|u\|_{0,\varepsilon}^2 \leq C \left\{ \sum_{|I| \leq r} \|G_I u\|_{0,\varepsilon-1}^2 + \|u\|_{0,\varepsilon-1}^2 \right\}$$

where G_I is a commutator of some of the G_j for $1 \leq j \leq n$. From Lemma 2.5 we deduce the same estimate with Q_I instead of G_I. Now

$$\|Q_I u\|_{0,\varepsilon-1}^2 = (E_{\varepsilon-1} Q_I u, E_{\varepsilon-1} Q_I u) = (Q_I^* E_{\varepsilon-1}^* E_{\varepsilon-1} Q_I u, u).$$

(All the integration by parts can be made since Q_I and $E_{\varepsilon-1}$ are tangential.) Now $T_{2\varepsilon-1} = Q_I^* E_{\varepsilon-1}^* E_{\varepsilon-1}$ belongs to $\mathrm{Op}(\Sigma_{\rho-2-|I|}^{2\varepsilon-1})$ and $\rho - 2 - |I| > 1$. We write

$$(T_{2\varepsilon-1} Q_I u, u) = (T_{2\varepsilon-1} Q_j Q_{I'} u, u) - (T_{2\varepsilon-1} Q_{I'} Q_j u, u) = ①-②$$

and using the symbolic calculus of paradifferential operator we get

$$|①| \leq C(\|Q_j u\|_{0,0} + \|Q_{I'} u\|_{2\varepsilon-1}^2)$$

and the same inequality for ②. Taking $\varepsilon \leq 1/2^r$, by induction we get inequality (2.8). (See [8] for more details.)

From inequality (2.8) we get for every $t \in \mathbf{R}$

$$(2.9) \qquad \|u\|_{0,t+\varepsilon}^2 + \sum_{j=1}^{2n} \|Q_j u\|_{0,t}^2 \leq C\{(|E_t Qv, E_t v|) + \|v\|_{0,t}^2\}$$

for every u in E, where $Q = D_n^2 + \tilde{Q}$.

We apply (2.5) and (2.8) to the function $E_t v$, where E_t is tangential of order t. The only term whose treatment is not obvious is

$$I = |([\tilde{Q}, E_t]v, E_t v)|.$$

Using Proposition 1.4 of [8] (i.e. $[\tilde{Q}, R] = \sum_{j=1}^{2n} R_j Q_j + R_0$ where R is a pseudo-differential operator of order r, R_j p.d.o of order r and $R_0 \in \mathrm{Op}(\Sigma_{\rho-3}^r)$) it is easy to see that for every $\delta > 0$ we can find $C_\delta > 0$ such that

$$|I| \le \delta \sum_{j=1}^{2n} \|Q_j u\|_{0,t}^2 + C_\delta \|u\|_{0,t}^2$$

which proves (2.3).

The next step in the proof is the following

(2.10)   Let v be in $H_{\mathrm{loc}}^{1,t-1}(\bar{\mathbf{R}}_+^n)$ such that $v|_{x_n=0} = 0$ and Qv is in $H_{\mathrm{loc}}^{0,t}(\bar{\mathbf{R}}_+^n)$ then v is in $H_{\mathrm{loc}}^{1,t+\varepsilon-1}(\bar{\mathbf{R}}_+^n)$.

To prove this claim we take δ in $]0,1]$, φ in $C^\infty(\bar{\mathbf{R}}_+^n)$ with $\mathrm{supp}\,\varphi \subset K$ where K is a compact in $\bar{\mathbf{R}}_+^n$, $\varphi_1 \in C_0^\infty(\bar{\mathbf{R}}_+^n)$, $\varphi_1 = 1$ on K and we set

$$v_\delta = T_\delta v = \varphi_1 (1 - \delta\Delta_{x'})^{-1}\varphi v$$

then $v_\delta \in H^{1,t+1}$, $v_\delta|_{x_n=0} = 0$ and $\mathrm{supp}\,v_\delta$ is contained in a compact of $\bar{\mathbf{R}}_+^n$, so we can apply inequality (2.9) to v_δ and deduce

$$\|v_\delta\|_{0,t+\varepsilon} + \sum_{j=1}^{2n} \|Q_j v_\delta\|_{0,t} \le C\{\|T_\delta Qv\|_{0,t} + \|[T_\delta, Q]v\|_{0,t} + \|v_\delta\|_{0,t}\}.$$

If we prove that $\|[T_\delta, Q]v\|_{0,t}$ is uniformly bounded in δ we shall have, letting δ go to zero, $v \in H^{0,t+\varepsilon}$, $D_n v \in H^{0,t} \subset H^{0,t+\varepsilon-1}$ so $v \in H^{1,t+\varepsilon-1}$. Now $Q = D_n^2 + \tilde{Q} + R$, where R denotes the lower order terms

$$[Q, T_\delta] = [D_n^2, T_\delta] + \sum_{j=1}^{n} \tilde{Q}^{(j)} T_{\delta(j)} - T_\delta^{(j)} \tilde{Q}_{(j)}$$

$$+ \sum_{2 \le |\alpha| \le [\rho-1]} (\tilde{Q}^{(\alpha)} T_{\delta(\alpha)} - T_\delta^{(\alpha)} \tilde{Q}_{(\alpha)}) + [R + R', T_\delta].$$

Now, since

$$\partial_\xi^\alpha (1 + \delta|\xi|^2)^{-1} \le C(1 + |\xi|^2)^{-|\alpha|/2}(1 + \delta|\xi|^2)^{-1}$$

with C independent of δ it is easy to see that

$$T_0 = \sum_{2 \le |\alpha| \le [\rho-1]} (\tilde{Q}^{(\alpha)} T_{\delta(\alpha)} - T_\delta^{(\alpha)} \tilde{Q}_{(\alpha)}) + [R + R', T_\delta]$$

can be written as $T_0 = \tilde{T}_0 T_\delta' + T_\delta^{-1}$ where T_δ' has the same form as T_δ, $\tilde{T}_0$ is of order zero uniformly bounded and T_δ^{-1} is of order -1, uniformly bounded with compact support.

On the other hand we also have (see [8]) with the same notations as above

$$T_\delta^{(j)} \tilde{Q}_{(j)} = \tilde{T}_0 Q_{n+j} T_\delta + \tilde{T}_0 T_\delta' + T_\delta^{-1}$$

so

$$||[\tilde{Q}, T_\delta]v||_{0,t}^2 \leq C\left\{\sum_{j=1}^{n} ||Q_j T_{\delta(j)}v||_{0,t}^2 + \sum_{j=1}^{n} ||Q_{n+j} T_\delta v||_{0,t}^2 \right.$$

$$\left. + ||T_\delta' v||_{0,t}^2 + ||T_\delta^{-1} v||_{0,t}^2 \right\}.$$

Now

$$\sum_{j=1}^{2n} ||Q_j T_{\delta(k)}v||_{0,t}^2 \leq C\{|(QE_t T_{\delta(k)}v, E_t T_{\delta(k)}v)| + ||T_{\delta(k)}v||_{0,t}^2\}.$$

A straightforward computation shows that the right-hand side of this inequality is bounded for any $\mu > 0$ by

$$C\left\{\mu \sum_{j=1}^{2n} ||Q_j T_{\delta(k)}v||_{0,t}^2 + C_\mu(||T_{\delta(kj)}v||_{0,t}^2 + ||T_{\delta(k)}v||_{0,t}^2 + ||T_\delta^{-1}v||_s^2)\right\}.$$

It follows, taking μ small enough that

$$||[\tilde{Q}, T_\delta]v||_{0,t}^2 \leq C\{||T_\delta'v||_{0,t}^2 + ||T_\delta^{-1}v||_{0,t}^2\}$$

where T_δ' has the same form as T_δ thus bounded in $H^{0,t}$ and T_δ^{-1} is of order -1 with compact support thus bounded in $H^{0,t}$. This proves (2.10).

2.3 *End of the proof of Theorem 0.1.* Let v be a solution of (2.1), $v \in C_{loc}^\rho$, $\rho > \text{Max}(5, r+3)$. It follows that v is in $C_{loc}^\rho(\bar{\mathbf{R}}_+^n) \cap H_{loc}^{1,1}(\bar{\mathbf{R}}_+^n)$. By Theorem 1.3, Propositions 2.1 and 2.3 we have $\chi^*v \in H_{loc}^{1,1}(\bar{\mathbf{R}}_+^n)$ and $(P^*)'(\chi^*v) \in H_{loc}^{1,1}(\bar{\mathbf{R}}_+^n)$. From (2.10) we deduce $\chi^*v \in H_{loc}^{1,1+\varepsilon}$ so by Theorem 1.3(iii) $v \in H^{1,1+\varepsilon}$. Iterating this argument we prove that $v \in H_{loc}^{1,+\infty}$ so $P'v \in H_{loc}^{0,+\infty}$. Now

$$D_n^2 v = f - \sum_{\substack{|\alpha| \leq 2 \\ \alpha_n < 2}} \Pi_{a_\alpha}' D^\alpha v$$

so $D_n^2 v$ is in $H_{loc}^{0,+\infty}$ from which we deduce $v \in H^{2,+\infty}$.

Now $D^\alpha v \in H_{loc}^{1,+\infty}$ if $\alpha_n < 2$ and by Proposition 1.7 in [7], $H^{1,+\infty}$ is an algebra. So we can write

(2.11) $$D_n^2 v = -G(x, v(x), Dv(x), \tilde{D}^2 v(x)) \in H^{1,+\infty}$$

therefore $v \in H_{loc}^{3,+\infty}$ so the right-hand side of (2.11) belongs to $H_{loc}^{2,+\infty}$ which implies $v \in H_{loc}^{4,+\infty}$ and so on, i.e. $v \in H_{loc}^{k,+\infty}$ for every $k \in \mathbf{N}$ so $v \in C^\infty$ near the origin. The proof is complete.

BIBLIOGRAPHY

1. S. Alinhac, *Paracomposition et opérateurs paradifférentiels*, Comm. Partial Differential Equations **11** (1986), 87–121.
2. J. M. Bony, *Calcul symbolique et propagation des singularités pour les e.d.p. nonlinéaires*, Ann. Sci. Ecole Norm. Sup. **14** (1981), 209–246.
3. M. Derridj, *Un problème aux limites pour une classe d'opérateurs du second ordre hypoelliptiques*, Ann. Inst. Fourier (Grenoble) **21** (1971), 99–148.

4. J. J. Kohn and L. Nirenberg, *Degenerate elliptic-parabolic equations of second order*, Comm. Pure Appl. Math. **20** (1967), 797–872.

5. Y. Meyer, *Remarques sur un théorème de J. M. Bony*, Rend. Circ. Mat. Palermo **1** (1981), 1–20.

6. O. A. Oleinik and E. V. Radkevitch, *Second order equations with non negative characteristic form*, Plenum Press, New York, 1973.

7. M. Sablé-Tougeron, *Régularité microlocale pour des problèmes aux limites nonlinéaires*, Ann. Inst. Fourier (Grenoble) **36** (1986).

8. C. J. Xu, *Régularité des solutions pour les e.d.p. quasi-linéaires nonelliptiques du second ordre*, Thèse Université Paris XI, 1984; C. R. Acad. Sci. Paris **300** (1985), 267–270.

9. _____ , *Opérateurs sous elliptiques et régularité des solutions des e.d.p. nonlinéaires du second ordre dans* $\mathbf{R}^2$, Comm. Partial Differential Equations **11** (1986), 1575–1603.

10. C. Zuily, *Sur la régularité des solutions non strictement convexes de l'équation de Monge-Ampère relle*, Prépublications d'Orsay 85 T.33 and article (to appear).

DEPARTMENT OF MATHEMATICS, WUHAN UNIVERSITY, WUHAN, HUBEI, PEOPLE'S REPUBLIC OF CHINA

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PARIS XI, ORSAY, CEDEX, FRANCE